Handout 1: The language of optimization: modeling definitions

In what follows are the key definitions for the course. You are responsible for any term below in the glossary that is in **boldface**. If you have questions about any definition, please consult your textbook and/or ask your instructor.

1 Modeling glossary

For the definitions listed, we assume n is a fixed, positive integer. We also use vector notation so that when we write $\mathbf{x} \in \mathbb{R}^n$, we mean that $\mathbf{x} = [x_1, \ldots, x_n]$, i.e., $\mathbf{x}$ is an n-dimensional vector with components $x_1, \ldots, x_n$.

1. **Constraint**: Any restriction, requirement or interaction that limits the values of the decision variables. Also called a **general constraint**. Some special constraints include:
   - **(Closed) linear constraint**: A constraint that can be written as a linear function on the decision variables that is set equal to or less than or equal to a constant. We typically omit stating "closed" which is the condition that only weak inequalities and equalities are allowed, e.g., $x < 5$ is **not** a (closed) linear constraint.
   - **Variable bounds**: An upper or lower bound on a single decision variable, e.g., $x \geq 0$ or $y \leq 3$ for decision variables x and y. *Note that a variable bound is also a linear constraint.*
2. **Decision variable**: An unknown quantity for an optimization model that represents a decision to be made. In this class we work with two kinds of decision variables (note these are constraints!):
   - **Continuous variable**: Decision variables constrained to be real numbers.
   - **Discrete variable**: Decision variables constrained to have values from a discrete set. E.g., $x \in \mathbb{Z}$ restricts x to the set of integers.
3. **Feasible region**: The set of allowable values for the decision variables.
4. **Feasible problem**: An optimization model is **feasible** if and only if a feasible solution exists for the problem.
5. **Feasible solution**: A solution to an optimization problem that satisfies all constraints.
6. **Integer program**: An optimization model where one or more of the decision variables must be integer.
7. **Linear function**: A function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if, and only if, there are constants $c_1, \ldots, c_n \in \mathbb{R}$ such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = c_1 x_1 + \ldots + c_n x_n.$$

8. **Linear program**: An optimization model where

(a) the decision variables are **continuous**;
(b) the objective function is **linear**; and
(c) there are a finite number of (closed) linear constraints, i.e., only equality and weak inequality linear constraints are permitted.

9. **(Model) constant**: A number in a model formulation that is fixed while solving. Also referred to as simply a constant.

10. **(Model) parameter**: A special kind of constant that is represented with a symbol and fixed while solving. Also referred to as simply a parameter.

11. **Nonlinear function**: A function that is not linear.

12. **Objective function**: For an optimization problem, a function on the decision variables to be maximized or minimized.

13. **Optimal solution**: For an optimization model with objective function f, a solution, $\mathbf{x} \in \mathbb{R}^n$ is optimal if and only if

    (a) $\mathbf{x}$ is feasible; and
    (b) for all feasible solutions $\mathbf{y}$, $f(\mathbf{x}) \geq f(\mathbf{y})$ for a maximization problem or $f(\mathbf{x}) \leq f(\mathbf{y})$ for a minimization problem.

14. **Optimization model**: Also called an **optimization problem** or a **mathematical program**, an optimization model consists of constants, an objective function, and possibly constraints. If the objective function is to be minimized then the problem is called a **minimization problem** and if the objective function is to be maximized then the problem is called a **maximization problem**. If the optimization problem has no constraints, we say it is an **unconstrained problem.** We say the optimization model is **over** $\mathbb{R}^n$ if there are n decision variables. Optimization models have one of four possible results:

    (a) **optimal**: A feasible problem where an optimal solution exists for the problem.
    (b) **infeasible**: No feasible solution exists for the problem.
    (c) **unbounded**: A feasible problem with objective function f, where for all real numbers, k, there exists a feasible solution $\mathbf{x} \in \mathbb{R}^n$ with $f(\mathbf{x}) \geq k$ for a maximization problem or $f(\mathbf{x}) \leq k$ for a minimization problem.
    (d) **unsolvable**: A feasible problem where no optimal solutions exist.
    Note that this result is **not** possible for a linear program!

15. **Parameterized optimization model**: An optimization problem where at least one constant is a parameter. We can write that a given optimization model is parameterized when it is a parameterized optimization model.

16. **Set**: A collection of objects, typically numbers, usually denoted by a comma separated list surrounded by curly braces and can be represented by a variable. E.g., The set, S, containing exactly the numbers 0, 4, and -2 could be written $S = \{0, 4, -2\}$. The symbol $\in$ is used to indicate whether an element is contained in a set. E.g., $0 \in S$ but $2 \notin S$. Two sets we use frequently are the set of integers, denoted by $\mathbb{Z}$, and the set of real numbers, denoted by $\mathbb{R}$.

17. **Solution**: For an optimization model over $\mathbb{R}^n$, a solution is a specific setting of decision variables.

18. **Value**: For an optimization model with objective function f, a value or **objective value** of a given solution $\mathbf{x} \in \mathbb{R}^n$, is the real number $f(\mathbf{x})$.

2 Examples

1. The following optimization problem is an unconstrained problem:

$$\max \;\; 2z.$$

   Here $n = 1$, the decision variable is z and the objective function is $f(z) = 2z$. There are no constraints. Note that the optimization model is a maximization problem and unbounded. The constant for the problem is the objective function coefficient 2. The optimization problem is not parameterized.

2. Consider the following optimization problem:

$$\max \;\; 2z \quad \text{subject to} \quad z \leq 1.$$

   Again, $n = 1$, the decision variable is z, the objective function is $f(z) = 2z$, and there is a single constraint, $z \leq 1$. Note that the optimization model is a maximization problem and has an optimal result as $z = 1$ is an optimal solution. The value of the optimal solution is 2. The constants for the problem are the objective coefficient 2, and the coefficient 1 on z and the right-hand side 1 of the constraint $z \leq 1$. The optimization problem is not parameterized. It is a linear program because the decision variable is continuous and the objective function and constraint are linear. The feasible region of the problem is the set $\{z \in \mathbb{R} : z \leq 1\}$ and the solutions $z = 1$, $z = 0$, $z = -4$ are all feasible. The solutions $z = 2$ and $z = 8$ are both infeasible.

3. The following optimization problem:

$$\max \;\; 2z \quad \text{subject to} \quad z < 1.$$

   Again, $n = 1$, the decision variable is z, the objective function is $f(z) = 2z$, and there is a single constraint, $z < 1$. Note that the optimization model is a maximization problem but is unsolvable as no optimal solution exists, since z can be arbitrarily close to 1, but not equal to 1. The constants for the problem are the objective coefficient 2, and the coefficient 1 on z and the right-hand side 1 of the constraint $z < 1$. The optimization problem is not parameterized and not a linear program because although the objective function is linear, the inequality in the constraint is NOT weak.

4. Let $c \in \mathbb{R}$ be a fixed constant. Then

$$\max \;\; 2z \quad \text{subject to} \quad z \leq c$$

   is a parameterized optimization model. Again, $n = 1$, the decision variable is z, the objective function is $f(z) = 2z$, and there is a single constraint, $z \leq c$. Note that the optimization model is a maximization problem with $z = c$ as an optimal solution so the problem has an optimal result. The value of the optimal solution is $2c$. The constants for the problem are the objective coefficient 2, and the coefficient 1 on z and the right-hand side c of the constraint $z \leq c$. The optimization problem is also a linear program because the decision variable is continuous, and the objective function and constraint are linear. As is often true for parameterized optimization models, because each $c \in \mathbb{R}$ represents a different optimization model, the formulation represents an infinite number of optimization problems.

5. Let $f(x, y) = 3x + y$, $g(x) = x^2 + y^2$, and $h(x) = 5x$.

   (a) The optimization problem over $\mathbb{R}^2$

   $$\max \quad f(x) \quad \text{subject to} \quad g(x) \leq 4$$

   is not a parameterized optimization model. The objective function is f, and the single constraint is $g(x) \leq 4$. The constants for this function are the coefficients 3 and 1 of f, the exponents 2 for each term in g, the coefficients 1 on both terms in g, and the right-hand side of 4 of the constraint $g(x) \leq 4$. An optimal solution for this maximization problem is $x = 2, y = 0$ with value 6, so the problem has an optimal result. The optimization model is not a linear program because g is not linear.

   (b) The optimization problem, again over $\mathbb{R}^2$,

   $$\min \quad g(x) \quad \text{subject to} \quad h(x) \leq 1 \quad \text{and} \quad h(x) \geq 2$$

   has objective function g, and two constraints $h(x) \leq 1$ and $h(x) \geq 2$. The constants for this function are the exponents 2 for each term in g, the coefficients 1 on both terms in g, the coefficient 5 in h, and the right-hand sides of 1 and 2 of the constraints $h(x) \leq 1$ and $h(x) \leq 2$, respectively. This minimization problem is infeasible because no $x \in \mathbb{R}$ exists that satisfies both $5x \leq 1$ and $5x \geq 2$. The optimization model is not a linear program because g is not linear.

6. Consider the following optimization problem.

$$\begin{array}{lll} \min & -x + 4y & \\ \text{subject to} & x + y & = 3 \\ & x - y + 5z & \geq 1 \\ & x & \leq 0. \end{array}$$

   The objective function is $-x + 4y$. The constraints are $x + y = 3$, $x - y + 5z \geq 1$, and $x \leq 0$. The constants are the coefficients -1 and 4 of the objective function, and the coefficients 1, 1, 3, 1, -1, 5, 1, 1, and 0 of the constraints. The optimization model is not parameterized. The optimization model is a linear program.

7. Consider the following optimization problem.

$$\begin{array}{lll} \min & -x + 4y & \\ \text{subject to} & x + y & = 3 \\ & x - y + 5z & \geq 1 \\ & x & \leq 0 \\ & y \in \{0, 1\}. & \end{array}$$

   This optimization model is the same as the one above, except with the additional constraint $y \in \{0, 1\}$. The optimization model is *not* a linear program as y is constrained to be either the integer 0 or 1. In fact, the optimization model is a (linear) integer program.

3 Homework problems

1. For each function on x, y, and z, indicate whether it is linear or not.

   (a) $f(x, y) = x + 5y - \log(x)$.

   (b) $f(x, y) = x + 5y$.

   (c) $f(x, y, z) = \pi x - \sqrt{3}y + \sin(\log(100))z$.

2. Let x, y, and z be decision variables and let a, b, and c be given parameters. Indicate whether each of the following sets of constraints could be in a linear program or not.

   (a) $x + 5y - z \leq 54$.

   (b) $x \geq abc, \log(a) \geq y \geq b, z \leq 5$.

   (c) $x > 0$.

   (d) $x^2 + y^2 \leq 9$.

   (e) $x + y \leq 2, x \in \{0, 1, 2\}$.

3. Consider the following optimization problem with decision variables x, y, and z.

$$\begin{array}{lll} \min & 3x + 2y - 4z & \\ \text{subject to} & 2x + 4y & \leq 4 \\ & \quad 3y + 2z & \leq 7 \\ & x & \geq 0. \end{array}$$

   What is the objective function? What are the constraints? Is the problem an parameterized optimization model? Is the problem a linear program or not? If not, indicate all the reasons the problem is not a linear program.

4. Consider the following optimization problem with decision variables x_1 and x_2.

$$\begin{array}{lll} \max & c_1x_1 + c_2x_2 & \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 & = b_1 \\ & a_{21}x_1 + a_{22}x_2 & = b_2 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0. \end{array}$$

What is the objective function? What are the constraints? Is the problem an parameterized optimization model? Is the problem a linear program or not? If not, indicate all the reasons the problem is not a linear program.

5. Consider the following optimization problem with decision variables x_1 and x_2.

$$\begin{array}{lll} \max & x_1 + 5x_2 & \\ \text{subject to} & x_1 + x_2 - x_1x_2 & = 5 \\ & a_{21}x_1^2 - 2x_2 & \leq 2 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0 \\ & x_1 \in \mathbb{Z}. & \end{array}$$

What is the objective function? What are the constraints? Is the problem a parameterized optimization model? Is the problem a linear program or not? If not, indicate all the reasons the problem is not a linear program.

6. Consider the following optimization problem with decision variables x_1 and x_2.

$$\begin{array}{lll} \max & cx_1 + 5x_2 & \\ \text{subject to} & x_1 + x_2 & = 5 \\ & a_{21}x_1^2 - 2x_2 & \leq 2 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0. \end{array}$$

What is the objective function? What are the constraints? Is the problem a parameterized optimization model? Is the problem a linear program or not? If not, indicate all the reasons the problem is not a linear program.